

## Dissipative Boussinesq system of equations in the Bénard-Marangoni phenomenon

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(Received 24 November 1992; revised manuscript received 19 October 1993)

By using the long-wavelength approximation, a system of coupled evolution equations for the bulk velocity and the surface perturbations of a Bénard-Marangoni system is obtained. It includes nonlinearity, dispersion, and dissipation, and it can be interpreted as a dissipative generalization of the usual Boussinesq system of equations. As a particular case, a strictly dissipative version of the Boussinesq system is obtained.

PACS number(s): 47.20.Bp, 47.35.+i

Several recent works [1–5] have dealt with the study of oscillatory instabilities in systems of the Rayleigh-Bénard and Bénard-Marangoni type. Considering a shallow fluid layer bounded below by a plane stress-free perfect conducting plate, and with a deformable upper free surface where a constant heat flux is imposed, the linear stability analysis indicates that oscillatory motion sets in when a certain critical combination of the Rayleigh and Marangoni numbers is attained [1]. On the other hand, it has been shown that, when the surface-tension effects can be disregarded, and the Rayleigh number of the system is at  $R = 30$ , appropriate surface deflections governed by the Korteweg-de Vries (KdV) equation may appear [2]. These results have been extended to the Bénard-Marangoni system [3,4], where buoyancy is discarded. In this case, appropriate surface disturbances propagate according to the KdV equation when the Marangoni number of the system is at  $M = -12$ . The physical mechanism behind such sustained excitations is the balance, at the critical point, between the energy dissipated by viscous forces and that released either by buoyancy or by a temperature-depending surface tension. In both cases, out of the critical points, appropriate surface excitations are governed by the Burgers equation [4,5].

The KdV equation is well known to govern long surface waves in inviscid shallow fluids [6]. It corresponds to a situation where nonlinearity and dispersion compensate each other, making possible the existence of coherent wave structures, like the solitary wave. The reductive perturbation method of Taniuti [7], based on the concept of stretching and in which waves propagating in only one direction are sought from the beginning, is a common approach to studying long waves in shallow water. From the various possible stretchings of the coordinates, different evolution equations may emerge as governing surface disturbances. However, there is an alternative approach to the theory of long waves in shallow water, which is based on perturbative expansions in two small parameters [6]. One of them measures the smallness of the amplitude perturbation, and the other is a measure of the longness of the wavelength perturbation. This approach, as an intermediate step, and from different possible relations between the two perturbative parameters,

yields different systems of evolution equations describing superimposed waves propagating in opposite directions. Only when specializing to waves moving in a given direction, do equations like breaking wave and KdV show up. A perturbative scheme of this kind has not been used to study surface excitations in Bénard systems. Consequently, for these systems, the more general evolution equations, wherefrom Burgers and KdV equations are obtained, have not been found.

In this paper, instead of using the reductive perturbation method of Taniuti [7], we will proceed through a perturbation scheme for the Bénard-Marangoni system based on two perturbative parameters, leading to the so-called long waves in shallow water approximation. By this way, a new system of coupled evolution equations will be found, involving the fluid velocity and the free-surface displacement. This system will be interpreted as a dissipative generalization of the Boussinesq equations. When the Marangoni number assumes the critical value  $M = -12$ , and a certain relation between the perturbative parameters is assumed, it reduces, as we are going to see, to the usual Boussinesq equations. A further restriction to waves moving in only one direction will lead to the KdV equation. On the other hand, out of the critical point, and assuming a different relation between the perturbative parameters, the dissipative generalization of the Boussinesq equations reduces to a strictly dissipative version of the Boussinesq equations. In this case, a restriction to waves moving in only one direction will lead to the Burgers equation.

We now turn to the description of the basic equations and boundary conditions. We consider a fluid bounded below by a plane, stress-free, perfect thermally conducting plate at  $z = 0$ , and above by a deformable one-dimensional free surface, which, at rest, lies at  $z = d$ . The depth  $d$  will be supposed to be small enough so that buoyancy can be neglected when compared to the effects coming from the surface tension dependence on temperature. In other words, we will be dealing with a Bénard-Marangoni system. The equations that describe such a system are

$$\nabla \cdot \mathbf{v} = 0, \quad (1)$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{g}\rho, \quad (2)$$

$$\frac{dT}{dt} = \kappa \nabla^2 T, \quad (3)$$

where  $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$  is the convective derivative,  $\mathbf{v} = (u, 0, w)$  is the fluid velocity, and  $p$  is the pressure. The density  $\rho$ , the viscosity  $\mu$ , and the thermal diffusivity coefficient  $\kappa$  are supposed to be constant. The surface tension  $\tau$  will be assumed to depend linearly on the temperature:

$$\tau = \tau_0 [1 - \gamma(T - T_0)], \quad (4)$$

where  $\gamma$  is a constant, and  $\tau_0, T_0$  are reference values for the surface tension and the temperature, respectively.

The boundary conditions on the upper free surface  $z = d + \eta(x, t)$  are [8]

$$\eta_t + u\eta_x = w, \quad (5)$$

$$(p - p_\alpha) - \frac{2\mu}{N^2} (w_z + u_x \eta_x^2 - \eta_x u_z - \eta_x w_x) = -\frac{\tau}{N^3} \eta_{xx}, \quad (6)$$

$$\mu (1 - \eta_x^2) (u_z + w_x) + 2\mu \eta_x (w_z - u_x) = N (\tau_x + \eta_x \tau_z), \quad (7)$$

$$\eta_x T_x - T_z = \frac{F}{k} N, \quad (8)$$

where  $F$  is the normal heat flux,  $k$  is the thermal conductivity,  $p_\alpha$  is a pressure exerted on the upper surface, all of them supposed to be constant, and  $N = (1 + \eta_x^2)^{\frac{1}{2}}$ . Subscripts denote partial derivatives with respect to the corresponding coordinate.

On the lower plane  $z = 0$ , we assume stress-free boundary conditions:

$$w = u_z = 0. \quad (9)$$

Moreover, we will assume that the lower plate is at a constant temperature  $T = T_b$ .

The static solution to the above equations is given by

$$p_s = p_\alpha - \rho g(z - d),$$

$$T_s = T_0 - \frac{F}{k}(z - d).$$

We consider now perturbations from this quiescent state. The horizontal and vertical length scales of these perturbations are supposed to be  $l$  and  $a$ , respectively. Then, we define two small parameters

$$\epsilon = \frac{a}{d}, \quad \delta, \quad (10)$$

which will be used to order the expansions. Before proceeding further, however, it is convenient to write the equations, boundary conditions, and static solutions in a

dimensionless form. This is done by taking the original variables (primed) to be

$$x' = lx, \quad z' = dz, \quad t' = \frac{l}{c_0} t, \quad (11a)$$

$$\eta' = a\eta, \quad u' = \frac{ag}{c_0} u, \quad w' = \frac{ag}{c_0 \delta} w, \quad (11b)$$

where  $c_0^2 = gd$ . Furthermore, four dimensionless parameters appear: the Prandtl number  $\sigma = \mu/\rho\kappa$ ; the Reynolds number  $R = c_0 d \rho/\mu$ ; the Bond number  $B = \rho g d^2/\tau_0$ , and the Marangoni number  $M = \gamma F d^2 \tau_0/k\kappa\mu$ .

To obtain the nonlinear evolution of the surface perturbations in the shallow water theory, we expand all variables in powers of  $z$ , keeping the terms that will contribute to the evolution equations up to orders  $\epsilon$  and  $\delta^2$ . Despite being laborious, this procedure is straightforward, and for this reason we will only give the guidelines, omitting the details of the calculations. To start with, we make the expansions

$$u = \sum_{n=0}^{\infty} z^n u_n, \quad w = \sum_{n=0}^{\infty} z^n w_n,$$

where  $u_n$  and  $w_n$  are both functions of  $x$  and  $t$ . Then, substituting them in Eq. (1), we get the relation

$$w_{n+1} = -\delta^2 \frac{u_{nx}}{n+1}.$$

Using the boundary conditions at  $z = 0$ , it is easy to show that

$$w_0 = w_2 = w_4 = \dots = 0,$$

$$u_1 = u_3 = u_5 = \dots = 0.$$

Then, using the expansion

$$p = p_s + \sum_{n=0}^{\infty} z^n p_n$$

in Eq. (2), it is possible to obtain  $u_2, u_4, u_6$ , and  $p_2$  in terms of  $u_0$  and  $p_0$  only. The other components of the expansions will contribute to orders higher than  $\epsilon$  and  $\delta^2$ , and therefore they can be neglected.

Next, expanding the temperature according to

$$T = T_s + \sum_{n=0}^{\infty} z^n \theta_n,$$

and using Eq. (3) with the corresponding boundary conditions, we can see that

$$\theta_0 = \theta_2 = \theta_4 = \theta_6 = \dots = 0.$$

Moreover, we can also obtain expressions for  $\theta_3$  and  $\theta_5$  in terms of  $u_0$  and  $p_0$  only. Now, Eq. (6) yields  $p_0$  in terms of  $u_0$  and  $\eta$ . Consequently, it is possible to rewrite the  $u$ 's,  $p$ 's, and  $\theta$ 's in terms of  $u_0$  and  $\eta$  only. Finally, using the above results in Eqs. (5) and (7), we obtain, up to order  $\epsilon$  and  $\delta^2$ , a coupled system of evolution equations for  $u_0$  and  $\eta$ :

$$u_{0t} + \epsilon u_0 u_{0x} + c^2 \eta_x - \frac{\delta}{R} \left(4 + \frac{M}{3}\right) u_{0xx} + \frac{\delta R}{6} (u_{0tt} + \eta_{xt}) + \frac{\delta^2 R^2}{120} (u_{0ttt} + \eta_{xtt}) - \delta^2 \left[ \frac{11}{6} - \frac{M}{30} (4\sigma - 1) \right] u_{0xxt} - \delta^2 \left( \frac{1}{B} + \frac{M}{30} + 1 \right) \eta_{xxx} = 0, \quad (12)$$

$$\eta_t + u_{0x} + \epsilon (u_0 \eta)_x + \frac{\delta R}{6} (u_{0xt} + \eta_{xx}) - \frac{\delta^2}{2} u_{0xxx} + \frac{\delta^2 R^2}{120} (u_{0xtt} + \eta_{xxt}) = 0, \quad (13)$$

where

$$c^2 = 1 - \frac{M}{\sigma R^2}. \quad (14)$$

The velocity  $u_0$  is only the first term in the expansion of  $u$ , which is

$$u = u_0 + \delta \frac{z^2 R^2}{2} \left( u_{0t} + \eta_x - \frac{3\delta}{R} u_{0xx} \right) + \delta^2 \frac{z^4 R^2}{24} (u_{0tt} + \eta_{xt}) + O(\epsilon \delta, \delta^3).$$

The value averaged over the depth is

$$\tilde{u} = u_0 + \delta \frac{R}{6} (u_{0t} + \eta_x) - \frac{\delta^2}{2} \left[ u_{0xx} - \frac{R^2}{60} (u_{0tt} + \eta_{xt}) \right] + O(\epsilon \delta, \delta^3).$$

The inverse is

$$u_0 = \tilde{u} - \delta \frac{R}{6} (\eta_x + \tilde{u}_t) + \frac{\delta^2}{2} \left[ \tilde{u}_{xx} + \frac{7R^2}{180} (\eta_{xt} + \tilde{u}_{tt}) \right] + O(\epsilon \delta, \delta^3).$$

Substituting this in Eqs. (12) and (13), and using the lowest-order equations in the  $\tilde{u}_{xxt}$  term, we obtain (omitting the tilde)

$$u_t + c^2 \eta_x + \epsilon u u_x - \frac{\delta}{R} \left(4 + \frac{M}{3}\right) u_{xx} + \delta^2 \Lambda \eta_{xtt} = 0, \quad (15a)$$

$$\eta_t + u_x + \epsilon (u\eta)_x = 0, \quad (15b)$$

where

$$\Lambda = \frac{4}{3} - \frac{1}{c^2} \left(1 + \frac{1}{B}\right) + \frac{M}{30} \left(1 - \frac{1}{c^2} - 4\sigma\right) + \frac{1}{6} \left(4 + \frac{M}{3}\right) \left(1 - \frac{1}{c^2}\right). \quad (16)$$

Due to the presence of the  $u_{xx}$  term in Eqs. (15), this system can be considered as a dissipative generalization of the Boussinesq equations. When the Marangoni number is at the critical value  $M = -12$ , these equations coincide with the usual Boussinesq system of equations

$$u_t + c^2 \eta_x + \epsilon u u_x + \delta^2 \Lambda \eta_{xtt} = 0, \quad (17a)$$

$$\eta_t + u_x + \epsilon (u\eta)_x = 0. \quad (17b)$$

In this case, by assuming that  $\delta^2 \approx \epsilon$ , and by specializing to waves moving, say, to the right, we can obtain, through a standard procedure [6], the KdV equation

$$\eta_t + c\eta_x + \epsilon \frac{3c}{2} \eta \eta_x + \delta^2 \frac{\Lambda}{2} \eta_{xxx} = 0, \quad (18)$$

where now

$$\Lambda = \frac{1}{5} \left( \frac{14}{3} + 8\sigma \right) - \frac{1}{c^2} \left( \frac{3}{5} + \frac{1}{B} \right). \quad (19)$$

Let us now consider the case  $M \neq -12$ . Assuming that  $\delta \approx \epsilon$ , and neglecting terms of order  $\delta^2 \approx \epsilon^2$ , Eq. (15) becomes

$$u_t + c^2 \eta_x + \epsilon u u_x - \frac{\delta}{R} \left(4 + \frac{M}{3}\right) u_{xx} = 0, \quad (20a)$$

$$\eta_t + u_x + \epsilon (u\eta)_x = 0. \quad (20b)$$

This is a strictly dissipative version of the Boussinesq system, since the dispersive term is not present now. By specializing again to waves moving to the right, we obtain the Burgers equation

$$\eta_t + c\eta_x + \epsilon \frac{3c}{2} \eta \eta_x - \frac{\delta}{2R} \left(4 + \frac{M}{3}\right) \eta_{xx} = 0. \quad (21)$$

We now summarize the results obtained in this paper. First, from a perturbative analysis, corresponding to the long wave in the shallow water approximation, we obtained a dissipative generalization of the Boussinesq system of equations as governing the bulk velocity and surface perturbations of a Bénard-Marangoni phenomenon. This system of coupled evolution equations includes nonlinearity, dispersion, and dissipation. The predominance of any one of them depends on the relation between  $\epsilon$  and  $\delta$ . Three cases are of special interest. First, when  $M = -12$ , the dissipative term vanishes, and assuming that  $\delta^2 \approx \epsilon$ , the usual Boussinesq system of equations is obtained. Second, when the Marangoni number is far enough from the critical value, that is, when  $M + 12 = O(1)$ , and assuming now that  $\delta \approx \epsilon$ , the dissipative term dominates the dispersive one. Then, by neglecting the dispersive term we get Eq. (20), which is a strictly dissipative version of the Boussinesq system of equations. And finally, as an intermediate case, we may also consider the situation in which  $M + 12 = O(\delta)$ . Assuming again that  $\delta^2 \approx \epsilon$ , it turns out that the dispersive and dissipative terms of Eq. (15) are of the same

order. In this case, no term is neglected, and the generalized Boussinesq system, Eq. (15), will govern the bulk velocity and surface perturbations of the Bénard-Marangoni system. Upon specialization to waves moving, say, to the right, each one of these three cases will lead to a single evolution equation for the surface displacement, which will be, respectively, the KdV, Burgers, and KdV-Burgers equations. Finally, we would like to call the attention to the appearance of a new equation, which is a dissipative generalization of the Boussinesq system, and which seems to be a nonintegrable equation. The usual Boussinesq system of equations, Eqs. (17), may be transformed into the *classical* Boussinesq equations, also known as dispersive long-wave equations [9]. These

equations have already been shown to be integrable [10]. The solutions to KdV and Burgers equations have also been extensively discussed in the literature [11]. However, the generalized Boussinesq system, as well as its strictly dissipative version, seems not to have been handled. The existence of analytical solutions, therefore, is still an open issue.

The authors would like to thank Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil, for partial support. One of the authors (M.A.M.) would also like to thank the Instituto de Física Teórica, UNESP, for the kind hospitality, and the CNPq for financial support.

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- [1] R. D. Benguria and M. C. Depassier, *Phys. Fluids A* **1**, 1123 (1989).
- [2] C. M. Alfaro and M. C. Depassier, *Phys. Rev. Lett.* **62**, 2597 (1989).
- [3] A. N. Garazo and M. G. Velarde, *Phys. Fluids A* **3**, 2295 (1991).
- [4] R. A. Kraenkel, J. G. Pereira, and M. A. Manna, *Phys. Rev. A* **46**, 4786 (1992).
- [5] R. A. Kraenkel, J. G. Pereira, and M. A. Manna, *Phys. Rev. A* **45**, 838 (1992).
- [6] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
- [7] T. Taniuti, *Suppl. Prog. Theor. Phys.* **55**, 1 (1974).
- [8] J. V. Wehausen and E. V. Laitone, in *Encyclopedia of Physics*, edited by S. Flügge (Springer, Berlin, 1960), Vol. 9.
- [9] L. F. J. Broer, *Appl. Sci. Res.* **31**, 377 (1975).
- [10] D. J. Kaup, *Prog. Theor. Phys.* **54**, 396 (1975).
- [11] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic Press, London, 1982).